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# AN EXTENSION OF THE KAHANE-KHINCHINE INEQUALITY IN A BANACH SPACE

BY

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ABSTRACT

We show that the geometric mean of the norm of a linear combination of the Steinhaus variables with "coefficients" in a Banach space is equivalent to the variance of the norm. This extends a result of Kahane, who established the corresponding inequality for the $L^p$ means.

## 0. Introduction

Let $\omega_1, \omega_2, \ldots$ denote the *Steinhaus variables*: independent identically distributed random variables, uniformly distributed on [0, 1]. A classical version of Khinchine's inequality states that for any $p > 0$ there exist $c_p, C_p > 0$ such that for any $x_1, \ldots, x_N \in \mathbf{C}$ we have

$$C_p \left\{ \mathscr{E} \left| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right|^p \right\}^{1/p} \geqq \left\{ \mathscr{E} \left| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right|^2 \right\}^{1/2}$$

$$\geqq c_p \left\{ \mathscr{E} \left| \sum_{j=1}^{N} e^{2\pi i \omega_j} x_j \right|^p \right\}^{1/p}. \tag{1}$$

(Here and below "$\mathscr{E}$" denotes "expected value".)

This inequality was considerably generalized by Kahane, who showed that if $B$ is a Banach space and $x_1, \ldots, x_N \in B$ then

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$$(2)\qquad \begin{aligned} C_p\left\{\mathscr{E}\left\|\sum_{j=1}^{N} e^{2\pi i\omega_j}x_j\right\|^p\right\}^{1/p} &\geqq \left\{\mathscr{E}\left\|\sum_{j=1}^{N} e^{2\pi i\omega_j}x_j\right\|^2\right\}^{1/2} \\ &\geqq c_p\left\{\mathscr{E}\left\|\sum_{j=1}^{N} e^{2\pi i\omega_j}x_j\right\|^p\right\}^{1/p}. \end{aligned}$$

(See Theorem 4 in Chapter 2 of [KH] or Exercise 1 on page 176 of [AG] for the original proof; a different argument due to C. Borell appears in Section 1.e of [LT], Volume II.) (Note that the middle term in (2) is *not* equivalent to $\{\Sigma_{j=1}^{N} \| x_j \|^2\}^{1/2}$ unless $B$ is isomorphic to a Hilbert space; see [KW].)

It was shown in [UK] that (1) extends to the case $p = 0$:

$$(3)\qquad \exp\left\{\mathscr{E}\log\left|\sum_{j=1}^{N} e^{2\pi i\omega_j}x_j\right|\right\} \geqq c\left\{\mathscr{E}\left|\sum_{j=1}^{N} e^{2\pi i\omega_j}x_j\right|^2\right\}^{1/2} \qquad (x_1,\ldots,x_n \in \mathbf{C}).$$

We should perhaps point out that the corresponding inequality with Rademacher functions in place of the Steinhaus variables is false, so that (3) cannot quite follow from the central limit theorem *per se*. On the other hand, (3) would be trivial if the Steinhaus variables were replaced by Gaussian variables, but the Gaussian version of (3) does not suffice for the applications given in [UK].

The purpose of the present paper is to prove a Banach-space version of (3) (Theorem 1 below).

The proof of (2) in [UK] is by the so-called "method of characteristic functions", also known as the Fourier transform. Having attempted to demonstrate the contrary, we must agree with the authors of [AG] that such methods seem to be of little use in proving results such as this in the infinite-dimensional case. However, it turns out that Kahane's proof of (2) can be adapted to the present case. The main step in the proof of (2) given in [KH] is a lemma stating roughly that if $\lambda > 0$ is such that our sum probably has norm not exceeding $\lambda$, then it is extremely unlikely that the norm exceeds $2\lambda$ ([KH], Theorem 3 (Chapter 2)). We prove a sort of "concentration inequality" in the other direction: There exist $\alpha > 0$ and $\gamma < 1$ such that, under suitable hypotheses, the probability that the norm of our sum is less than $\alpha\lambda$ does not exceed $\gamma$ times the probability that the norm is less than $\lambda$. See Lemmas 1 and 2 below.

We wish to thank A. Pelczyński for suggesting the question.

NOTE. Recall that a Banach space is said to have *cotype 2* if

$$(4) \qquad \sum_{j=1}^{N} \| x_j \|^2 \leqq c\mathscr{E} \left\| \sum_{j=1}^{N} e^{2\pi i\omega_j} x_j \right\|^2 .$$

(This is equivalent to the usual definition in terms of Rademacher functions; see the Appendix to [AG].) Thus our theorem has as a corollary

$$(5) \qquad \exp \mathscr{E} \log \left\| \sum_{j=1}^{N} e^{2\pi i\omega_j} x_j \right\| \geqq c \left\{ \sum_{j=1}^{N} \| x_j \|^2 \right\}^{1/2},$$

*if* $B$ has cotype 2. Now, if $B$ is a Banach *lattice* of cotype 2, then it is relatively easy to establish (5) using the Krivine functional calculus and the one-dimensional result in [UK]. (See [LT], Volume II, p. 42.) One might think that a proof of (5) in a lattice of cotype 2 "must" conceal a proof of Theorem 1 below in an arbitrary lattice. (How can $A \leqq B$ imply $A \leqq C$ unless $B \leqq C$?) But we have been unable to prove Theorem 1 in a lattice from the Krivine calculus. (Note that a proof of Theorem 1 in an arbitrary Banach lattice would give Theorem 1 in an arbitrary Banach space, since any Banach space is isometric to a subspace of some $C(K)$.)

Note. Our theorem immediately implies an apparently stronger statement concerning two (or finitely many, by induction) mutually independent sequences of Steinhaus variables; see the corollary in Section 3. This is one reason we wished to extend the one-dimensional inequality in [UK] to a vector-valued context: If one takes $B = \mathbf{C}$ in the corollary one obtains an extension of the inequality in [UK] the statement of which mentions nothing but scalars, but the proof of which appears to require our vector-valued inequality.

## 1. Theorem

We intend to state and prove our theorem *assuming* the validity of the lemmas to be presented in the next section.

Theorem. *There exists* $c > 0$ *such that if* $x_1, \ldots, x_N$ *are elements of the Banach space* $B$ *then*

$$(6) \qquad \exp \mathscr{E} \log \left\| \sum_{j=1}^{N} e^{2\pi i\omega_j} x_j \right\| \geqq c \left\{ \mathscr{E} \left\| \sum_{j=1}^{N} e^{2\pi i\omega_j} x_j \right\|^2 \right\}^{1/2} .$$

Proof. Let $x_1, \ldots, x_N \in B$, and let $\mu$ be the distribution (or the "law") of $\sum_{j=1}^{N} e^{2\pi i\omega_j} x_j$, so that

$$\mu(A) = P\left\{\sum_{j=1}^{N} e^{2\pi i\omega_j}x_j \in A\right\}.$$

Let $B_\lambda = \{x \in B : \|x\| < \lambda\}$. As in [UK], we need only show that

$$\int_0^1 \mu(B_\lambda)\lambda^{-1}d\lambda \leqq c \tag{7}$$

under the assumption (which we now assume) that $\mathscr{E}\,\|\Sigma_{j=1}^N e^{2\pi i\omega_j}x_j\|^2 = 1$. Suppose $\|x_1\| \geqq \|x_j\|$ for all $j$. The piece of the integral in (7) corresponding to $\lambda < \frac{1}{2}\|x_1\|$ can be handled by the same simple argument as in [UK]: If $\lambda < \frac{1}{2}\|x_1\|$ then for any $y \in B$ whatsoever the triangle inequality shows that

$$P\{\|e^{2\pi i\omega_1}x_1 - y\| < \lambda\} \leqq c\lambda/\|x_1\|,$$

so that independence gives

$$P\left\{\left\|\sum_{j=1}^{N} e^{2\pi i\omega_j}x_j\right\| < \lambda\right\} \leqq c\lambda/\|x_1\| \qquad (\lambda < \tfrac{1}{2}\|x_1\|).$$

Thus

$$\int_0^{\|x_1\|/2} \mu(B_\lambda)\lambda^{-1}d\lambda \leqq c.$$

Let $K$, $\delta$, and $\gamma$ be as in Lemmas 1 and 2. The above inequality shows that

$$\int_0^{2K\|x_1\|} \mu(B_\lambda)\lambda^{-1}d\lambda \leqq c. \tag{8}$$

Next we will show that

$$\mu(B_{\alpha\lambda}) \leqq \gamma\mu(B_\lambda) \qquad (2K\|x_1\| \leqq \lambda \leqq 1), \tag{9}$$

where $\alpha = \delta/2K$. Fix $\lambda \in [2K\|x_1\|, 1]$, and now pick an integer $M$ which is *maximal* subject to the condition

$$\left\{\mathscr{E}\left\|\sum_{j=1}^{M} e^{2\pi i\omega_j}x_j\right\|^2\right\}^{1/2} \leqq \lambda/K. \tag{10}$$

(We may certainly suppose $K > 1$, so that $\lambda/K < 1$, which implies that (10) fails if $M$ is large enough, since $\mathscr{E}\,\|\Sigma_{j=1}^N e^{2\pi i\omega_j}x_j\|^2 = 1$.)

Let $X = \Sigma_{j=1}^M e^{2\pi i\omega_j}x_j$ and $Y = \Sigma_{j=M+1}^N e^{2\pi i\omega_j}x_j$; let $\sigma^2 = \mathscr{E}\,\|X\|^2$. Then (10) states precisely that $K\sigma \leqq \lambda$. On the other hand, the triangle inequality and the maximality of $M$ show that $\sigma + \|x_{M+1}\| > \lambda/K$; since $\|x_1\| \geqq \|x_{M+1}\|$ and $\lambda \geqq 2K\|x_1\|$ this shows that that $\alpha\lambda = \delta\lambda/2K < \delta\sigma$. Now Lemma 2 gives (9):

$$\begin{aligned} P\{ \| X+Y \| < \alpha\lambda \} &\leqq P\{ \| X+Y \| < \delta\sigma \} \\ &\leqq \gamma P\{ \| X+Y \| < K\sigma \} \\ &\leqq \gamma P\{ \| X+Y \| < \lambda \}. \end{aligned}$$

(In other words: Although the distribution of our sum is presumably not infinitely divisible, it is pretty darn divisible — enough so to derive (9) from Lemma 2.)

Now we can conclude the proof:

Let $I_n = [\alpha^{n+1}, \alpha^n]$. Pick $n_0$ so $2K \| x_1 \| \in I_{n_0}$. Repeated use of (9) shows that $\mu(B_\lambda) \leqq \gamma^n$ for $\lambda \in I_n$, $n = 0, 1, \ldots, n_0$. Thus

$$\int_{\alpha^{(n_0+1)}}^{1} \mu(B_\lambda)\lambda^{-1} d\lambda \leqq \sum_{n=0}^{n_0} \gamma^n \log \frac{1}{\alpha} \leqq c. \tag{11}$$

Since $2K \| x_1 \| \geqq \alpha^{(n_0+1)}$, (8) and (11) give (7). QED, assuming Lemma 2.

## 2. Lemmas

In this section we shall state and prove Lemma 1, leaving it to the reader to deduce Lemma 2.

LEMMA 1. *There exist constants $\delta > 0$, $K < \infty$, and $\gamma \in (0, 1)$ such that if $x_1, \ldots, x_N \in B$, $X = \sum_{j=1}^N e^{2\pi i \omega_j} x_j$, and $\mathscr{E} \| X \|^2 = 1$, then*

$$P\{ \| X - y \| < \delta \} \leqq \gamma P\{ \| X - y \| < K \} \tag{12}$$

*for any $y \in B$.*

LEMMA 2. *There exist constants $\delta > 0$, $K < \infty$, and $\gamma \in (0, 1)$ such that if $x_1, \ldots, x_N \in B$, $X = \sum_{j=1}^N e^{2\pi i \omega_j} x_j$, and $\sigma^2 = \mathscr{E} \| X \|^2$ then*

$$P\{ \| X + Y \| < \delta\sigma \} \leqq \gamma P\{ \| X + Y \| < K\sigma \}$$

*for any $B$-valued random variable $Y$ which is independent of $X$.*

PROOF OF LEMMA 1. The result of Kahane referred to above shows that there exists $\beta > 0$ such that $\mathscr{E} \| X \| \geqq \beta \{ \mathscr{E} \| X \|^2 \}^{1/2} = \beta$. For $\varepsilon > 0$ define the event $A_\varepsilon = \{ \| X \| < \varepsilon \}$. Clearly a lower bound on $\mathscr{E} \| X \|$ together with an upper bound on $\mathscr{E} \| X \|^2$ show that $\| X \|$ cannot be too small on too large a set; in fact one sees that

$$P(A_\varepsilon) \leqq \frac{1-\beta^2}{1-2\varepsilon}. \tag{13}$$

(Cf. the "Paley–Zygmund inequalities" in [KH].)

Pick $\delta_0 > 0$ so that $(1-\beta^2)/(1-2\delta_0) < 1$, and let $p_0 = (1-\beta^2)/(1-2\delta_0)$. We will prove (12) with

$$\delta = \tfrac{1}{2}\delta_0, \quad K = \max\left(3\delta, \delta + \left(\frac{2}{1-p_0}\right)^{1/2}\right), \quad \text{and} \quad \gamma = \max\left(\frac{1}{2}, \frac{2p_0}{p_0+1}\right).$$

*Case* 1: $\| y \| \leqq \delta$. In this case

$$P\{\| X - y \| < \delta\} \leqq P\{\| X \| < 2\delta = \delta_0\} \leqq p_0$$

by (13), while $\mathscr{E} \| X \|^2 = 1$ shows that

$$P\{\| X - y \| < K\} \geqq P\{\| X \| < K - \delta\} \geqq 1 - (K-\delta)^{-2}.$$

This gives (12), since

$$\frac{p_0}{1-(K-\delta)^{-2}} \leqq \gamma.$$

*Case* 2: $\| y \| > \delta$. In this case we may pick a real number $\theta$ such that $|1 - e^{i\theta}| = 2\delta / \| y \|$. Let $\tilde{y} = e^{i\theta}y$; then $\| y - \tilde{y} \| = 2\delta$. For $z \in B$ and $r > 0$ define $B(z, r) = \{x \in B : \| z - x \| < r\}$. Now $\| y - \tilde{y} \| = 2\delta$ shows that $B(y, \delta)$ and $B(\tilde{y}, \delta)$ are disjoint, while symmetry shows that $P\{X \in B(\tilde{y}, \delta)\} = P\{X \in B(y, \delta)\}$. Since both are contained in $B(y, 3\delta)$, we see that

$$\begin{aligned} P\{\| X - y \| < K\} &\geqq P\{\| X - y \| < 3\delta\} \\ &\geqq P\{\| X - y \| < \delta\} + P\{\| X - \tilde{y} \| < \delta\} \\ &= 2P\{\| X - y \| < \delta\}. \end{aligned}$$

This gives (12), since $\gamma \geqq \frac{1}{2}$.

## 3. Corollary

Let $\tilde{\omega}_1, \tilde{\omega}_2, \ldots$ denote another sequence of Steinhaus variables, independent of $\omega_1, \omega_2, \ldots$ .

COROLLARY. *There exists $c > 0$ with the following property: If $B$ is a Banach space and $x_{j,k} \in B$ for $1 \leqq j, k \leqq N$ then*

$$\exp \mathscr{E} \log \left\| \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \right\| \geqq c \left\{ \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \left\| \right.^2 \right\}^{1/2}. \tag{14}$$

PROOF. Let $\Omega$ and $\tilde{\Omega}$ denote the probability spaces on which $\omega_j$ and $\tilde{\omega}_k$, respectively, are defined. Let $X = L_B^2(\tilde{\Omega})$ denote the space of square-integrable $B$-valued functions defined on $\tilde{\Omega}$. For $j = 1, \ldots, N$ let

$$\sum_{k=1}^{N} e^{2\pi i \tilde{\omega}_k} x_{j,k} = y_j \in X.$$

Now two applications of the theorem, once in $B$ and once in $X$, show that

$$\begin{aligned}
\exp \mathscr{E} \log & \left\| \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \right\|_B \\
&= \exp \mathscr{E}_\omega \log \exp \mathscr{E}_{\tilde{\omega}} \log \left\| \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \right\|_B \\
&\geqq c \exp \mathscr{E}_\omega \log \left\{ \mathscr{E}_{\tilde{\omega}} \left\| \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \right\|_B^2 \right\}^{1/2} \\
&= c \exp \mathscr{E}_\omega \log \left\| \sum_{j=1}^{N} e^{2\pi i \omega_j} y_j \right\|_X \\
&\geqq c \left\{ \mathscr{E}_\omega \left\| \sum_{j=1}^{N} e^{2\pi i \omega_j} \right\|_X^2 \right\}^{1/2} \\
&= c \left\{ \mathscr{E} \left\| \sum_{j,k=1}^{N} e^{2\pi i \omega_j} e^{2\pi i \tilde{\omega}_k} x_{j,k} \right\|_B^2 \right\}^{1/2}.
\end{aligned}$$

Q.E.D.

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